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Accuracy of Third-Order Predictor-Corrector Difference Schemes for Hyperbolic Systems

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THIS Note deals with numerical methods for the following system of equations:

$$\partial\phi/\partial t + \partial f(\phi, x, t)/\partial x = 0 \tag{1}$$

where $\phi^T = (\phi_1, \phi_2, \ldots, \phi_n)$, $f^T = (f_1, f_2, \ldots, f_n)$. The system is assumed to be hyperbolic. Equation (1) is of great importance in fluid mechanics, because both the equations of inviscid gas dynamics and the advection operator are of this form.

Difference schemes for the numerical solution of Eq. (1) should preferably be of predictor-corrector type, because then the matrix $F = \partial f/\partial\phi$ and its derivatives do not appear in the difference scheme. This eliminates time-consuming matrix multiplications. An often-used predictor-corrector scheme for the solution of Eq. (1) is the Lax-Wendroff scheme, as formulated by Richtmyer.[1] A third-order predictor-corrector scheme has been constructed by Rusanov[2] and by Burstein and Mirin.[3] This difference scheme derives special significance from the fact that it can be shown,[4] that there exists no other third-order predictor-corrector scheme for Eq. (1).

The Rusanov-Burstein-Mirin (RBM) scheme contains two parameters which may vary. One is the fraction $\tau\Delta t$ of the time-step Δt at which the first predictor is evaluated. The other is the damping coefficient ω. The parameter τ does not influence the amplification matrix (for a definition see Ref. 1), and therefore has only little influence on the accuracy. We will take $\tau = \frac{1}{3}$. The parameter ω has a large influence on the accuracy, as will be demonstrated. Furthermore, it will be shown that it is possible to improve the accuracy of the RBM scheme considerably by modifying the damping term.

The RBM scheme is defined as follows:

$$\phi_{j+1/2}^{(1)} = \tfrac{1}{2}(\phi_{j+1}^n + \phi_j^n) - (\sigma/3)(f_{j+1}^n - f_j^n) \tag{2}$$

$$\phi_j^{(2)} = \phi_j^n - \tfrac{2}{3}\sigma(f_{j+1/2}^{(1)} - f_{j-1/2}^{(1)}) \tag{3}$$

$$\phi_j^{n+1} = \phi_j^n - (\sigma/24)(-2f_{j+2}^n + 7f_{j+1}^n - 7f_{j-1}^n + 2f_{j-2}^n) - \tfrac{3}{8}\sigma(f_{j+1}^{(2)} - f_{j-1}^{(2)}) + D \tag{4}$$

$$D = -\tfrac{1}{24}\omega\delta^4\phi_j^n \tag{5}$$

where $\sigma = \Delta t/\Delta x$ and $\delta^4\phi_j^n$ is an undivided fourth difference. Equations (2)–(4) are called the first predictor, the second predictor, and the corrector, respectively. The quantity D is called the damping term. For stability it is necessary that the damping-coefficient ω satisfy

$$c_m^2(4 - c_m^2) \leqq \omega \leqq 3 \tag{6}$$

where the Courant number c_m is defined as $c_m = \sigma|\lambda_m|$, with λ_m the absolutely largest eigenvalue of F in the point $(n\Delta t, j\Delta x)$.

In order to exhibit the influence of ω on the accuracy, a model problem with one unknown is studied, with $f = u\phi$ $u = (a + b\cos^2\pi x)^{-1}$. For the determination of the errors in th numerical calculations use is made of the fact that with $\phi(0, x$ periodic in x with period 1 the exact solution is periodic in with period $a + b/2$.

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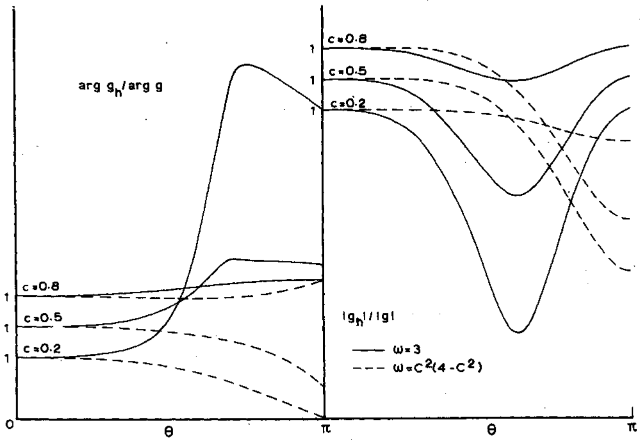

Fig. 1 Dispersion and dissipation of the Rusanov-Burstein-Miri scheme.

The following four cases were calculated: case 1: $a = b = 1$ $\phi(0, x) = H(x - \frac{1}{2})$; case 2: $a = b = 1$, $\phi(0, x) = \sin^2\pi x$; case $a = 1.05$, $b = 1.9$, $\phi(0, x) = H(x - \frac{1}{2})$; case 4: $a = 1.05$, $b = 1.9$ $\phi(0, x) = \sin^2\pi x$; where $H(x)$ is the periodic step function wit period 1: $H(x) = 0$ for $2m - 1 < x \leqq 2m$, $H(x) = 1$ for $2m < x \leqq 2m + 1$, with m an arbitrary integer. Table 1 gives for tw values of ω the average error ε, defined as

$$\varepsilon = \Delta x \sum_{j=0}^{j=1/\Delta x} |\phi_j^n - \phi(n\Delta t, j\Delta x)| \tag{7}$$

where ϕ is the exact (analytical) solution.

Table 1 shows that the accuracy depends strongly on th damping coefficient ω, especially when the solution is smootl This can be explained by a study of the difference between th amplification matrix of the differential equation and the ampl fication matrix of the difference scheme. With only one unknow these matrices reduce to factors g and g_h, respectively. Th amplification factor is the factor by which a harmonic wa $\exp(i\theta x/\Delta x)$ is multiplied when it is propagated over a tin interval Δt by the differential equation or the difference schem Figure 1 represents the quantities $|g_h/g|$ and $\arg g_h/\arg g$ functions of θ, for several values of the Courant number $c = \sigma$ These quantities will be called the dissipation and dispersic respectively. The figure shows that the amplification factor g_h strongly influenced by the damping coefficient ω, so that it is n surprising that ω has a large influence on the accuracy of t RBM scheme. Long waves, with little dispersion, are damp more strongly with $\omega = 3$ than with $\omega = c^2(4 - c^2)$. On the oth hand, short waves which have much dispersion for both values ω are damped strongly with $\omega = c^2(4 - c^2)$, but weakly or not all with $\omega = 3$. Clearly, with $\omega = c^2(4 - c^2)$ the RBM sche has much better dissipation and dispersion properties than w $\omega = 3$.

The question arises of whether there exists a better choice ω than $\omega = c^2(4 - c^2)$. This is probably not the case, beca the value $c^2(4 - c^2)$ is singled out by the fact, that with u constant the RBM scheme is fourth order accurate in x rat than third order.

When the number of unknowns is greater than one assumption of hyperbolicity implies the existence of a matri with the property $UFU^{-1} = \Lambda$, with Λ a diagonal matrix. the study of dissipation and dispersion F is assumed to constant. The diagonalized system then consists of n mutu independent equations, each with one unknown. The differe scheme for the kth unknown has the following damping term

$$D = -\tfrac{1}{24}\omega\delta^4\psi_{k,j}^n$$

Table 1 Average error for test cases 1–4

	Case 1, $t = 1.5$		Case 2, $t = 1.5$		Case 3, $t = 2$		Case 4, $t = 2$	
$\Delta x, \Delta t$	0.025	0.0625	0.025	0.0625	0.025	0.0625	0.025	0.0625
$\omega = 3$	0.1248	0.2331	0.0146	0.0819	0.1820	0.2466	0.0609	0.1851
$\omega = c^2(4-c^2)$	0.1019	0.1972	0.0029	0.0380	0.1411	0.2618	0.0215	0.1183

Table 2 Definition of test cases 5–8

Case	$\phi_1(0,x)$	$\phi_2(0,x)$	a_1	b_1	a_2	b_2
Case 5	$H(x-\frac{1}{2})$	$\cos^2 \pi(x-\frac{1}{4})$	1.05	0.9	2.5	1
Case 6	$\sin^2 \pi x$	$\cos^2 \pi(x-\frac{1}{4})$	1.05	0.9	2.5	1
Case 7	$H(x-\frac{1}{2})$	$\cos^2 \pi(x-\frac{1}{4})$	1.4	0.2	2.8	0.4
Case 8	$\sin^2 \pi x$	$\cos^2 \pi(x-\frac{1}{4})$	1.4	0.2	2.8	0.4

Table 3 Average error for test cases 5–8

Damping term		Case 5, $t = 3$		Case 6, $t = 3$		Case 7, $t = 3$		Case 8, $t = 3$	
	$\Delta x, \Delta t$	0.025	0.0625	0.025	0.0625	0.025	0.0625	0.025	0.0625
Eq. (5)	ε_1	0.1056	0.2241	0.0023	0.0383	0.0913	0.1868	0.00095	0.0118
	ε_2	0.0268	0.0405	0.0038	0.0428	0.0322	0.0390	0.00107	0.0155
Eq. (10)	ε_1	0.1032	0.2157	0.0025	0.0394	0.0610	0.1866	0.000092	0.0036
	ε_2	0.0314	0.0663	0.0022	0.0288	0.0150	0.0303	0.000081	0.0027

where ψ_k is the kth component of the vector $\psi = U\phi$. For optimum accuracy ω should satisfy $\omega = c_k^{\,2}(4-c_k^{\,2})$, with $c_k = \sigma|\lambda_k|$, where λ_k is the kth eigenvalue of F. However, stability requires $\omega \geqq c_m^{\,2}(4-c_m^{\,2})$, which is larger than $c_k^{\,2}(4-c_k^{\,2})$, except for the value of k which corresponds to the absolutely largest eigenvalue. The $(n-1)$ unknowns with a different value of k are evaluated with possibly strongly diminished accuracy.

However, it is possible to obtain optimum accuracy for all components simultaneously, if the scalar ω is replaced by the matrix Ω, defined as

$$\Omega = C^2(4I - C^2) \tag{9}$$

with $C = \sigma\Lambda$ and I the identity matrix. The damping term in Eq. (5) is then found to be

$$D = -\tfrac{1}{24}\delta^4 d_j^{\,n}, \qquad d = U^{-1}\Omega U\phi \tag{10}$$

Replacement of Eq. (5) by Eq. (10) conserves the third order accuracy of the RBM scheme, because the difference between Eqs. (5) and (10) is of fourth order. The predictor-corrector form is also preserved, because the matrix multiplications necessary to obtain d may be done analytically beforehand. For this the eigenvalues and the diagonalizing matrix of F are needed. These may be considered known, because in applications they are necessary for a good understanding of the phenomenon under study. Replacement of Eq. (5) by Eq. (10) makes the RBM scheme slightly more time-consuming, because in addition to the function evaluations necessary to determine f^n, $f^{(1)}$, and $f^{(2)}$ an additional function evaluation to determine d is necessary.

The considerable increase in accuracy which one may expect from replacing Eq. (5) by Eq. (10) is demonstrated by the following example:

$$F = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix}, \quad \begin{aligned} c_{1,2} &= (\lambda_1 \pm \lambda_2)/2 \\ \lambda_{1,2} &= (a_{1,2} + b_{1,2}\cos^2 \pi x)^{-1} \end{aligned} \tag{11}$$

If $\phi(0,x)$ is periodic in x with period 1 and $a_1 + \frac{1}{2}b_1 = p$, $a_2 + \frac{1}{2}b_2 = np$, where n is an integer, the exact solution is periodic in t with period np.

The four cases that were calculated are listed in Table 2. The average error in ϕ_1 and ϕ_2, called ε_1 and ε_2, respectively, is defined as in Eq. (7) and listed in Table 3. The results confirm that with the damping term D defined by Eq. (10) the RBM scheme is considerably more accurate than with D defined by Eq. (5), especially when the solution is smooth.

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An Automated Gradient Projection Algorithm for Optimal Control Problems

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Introduction

ONE method of treating optimal control problems with terminal state constraints is to adjoin these constraints to

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